

Holographic Baryons¹

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Abstract

We review baryons in the D4-D8 holographic model of low energy QCD, with the large N_c and the large 't Hooft coupling limit. The baryon is identified with a bulk soliton of a unit Pontryagin number, which from the four-dimensional viewpoint translates to a modified Skyrmion dressed by condensates of spin one mesons. We explore classical properties and find that the baryon in the holographic limit is amenable to an effective field theory description. We also present a simple method to capture all leading and subleading interactions in the $1/N_c$ and the derivative expansions. An infinitely predictive model of baryon-meson interactions is thus derived, although one may trust results only for low energy processes, given various approximations in the bulk. We showcase a few comparisons to experiments, such the leading axial couplings to pions, the leading vector-like coupling, and a qualitative prediction of the electromagnetic vector dominance that involves the entire tower of vector mesons.

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1 Low Energy QCD and Solitonic Baryons

QCD is a challenging theory. Its most interesting aspects, namely the confinement of color and the chiral symmetry breaking, have defied all analytical approaches. While there are now many data accumulated from the lattice gauge theory, the methodology falls well short of giving us insights on how one may understand these phenomena analytically, nor does it give us a systematic way of obtaining a low energy theory of QCD below the confinement scale.

A very useful approach in the conventional field theory language is the chiral perturbation theory [1]. It bypasses the question of how the confinement and the symmetry breaking occur but rather focuses on the implications. A quark bilinear condenses to break the chiral symmetry $U(N_F)_L \times U(N_F)_R$ to its diagonal subgroup $U(N_F)$, where by N_F^2 Goldstone bosons appear, which we will refer to as pions. They are singled out as the lightest physical particles, and one guesses and constrains an effective Lagrangian for them. In the massless limit^{#1} of the bare quarks, the pions are packaged into a unitary matrix as

$$U(x) = e^{2i\pi(x)/f_\pi} , \quad (2)$$

whose low energy action is written in a derivative expansion as

$$\int d^4x \left(\frac{f_\pi^2}{4} \text{tr} (U^{-1} \partial_\mu U)^2 + \frac{1}{32e_{\text{Skyrme}}^2} \text{tr} [U^{-1} \partial_\mu U, U^{-1} \partial_\nu U]^2 + \dots \right) , \quad (3)$$

where the ellipsis denotes higher derivative terms as well as other possible quartic derivative terms. One can further add other massive mesons whose masses and interaction strengths are all left as free parameters to fit with data.

Another analytical approach is the large N_c expansion [2]. Here, two different couplings $1/N_c$ and $\lambda = g_{YM}^2 N_c$ control the perturbation expansion, one counting the topology of the Feynman diagram and the other counting loops. An interesting question is how this large N_c limit appears in the chiral Lagrangian approach. Since the pion fields (or any other meson fields that one can add) are already color-singlets, N_c would enter only via the numerical coefficients of the various terms in the Lagrangian. Both terms shown in (18) can arise from planar diagrams of large N_c expansion, and we expect

$$f_\pi^2 \sim N_c \sim \frac{1}{e_{\text{Skyrme}}^2} . \quad (4)$$

^{#1} The effect of small bare masses for quarks can be incorporated by an explicit symmetry breaking term

$$\text{tr} (MU + U^\dagger M^\dagger) \quad (1)$$

with a matrix M , which in our holographic approach would be ignored.

Note that since $1/f_\pi^2$ and $1/(e_{Skyme}^2 f_\pi^4)$ play the role of squared couplings for canonically normalized pions, the self-coupling of pions scales as $N_c^{-1/2}$ [3]. In particular, this shows that baryons are qualitatively different than mesons in the large N_c chiral perturbation theory. Baryons involves N_c number of quarks, so the mass is expected to grow linearly with N_c , or equivalently grows with the inverse square of pion self-couplings. In field theories, such a scaling behavior is a hallmark of nonperturbative solitons.

Indeed, it has been proposed early on that baryons are topological solitons, namely Skyrmons [4], whose baryon number is cataloged by the third homotopy group of $U(N_F)$, $\pi_3(U(N_F)) = Z$. The topological winding is counted by how many times $U(x)$ covers a noncollapsible three-sphere in $U(N_F)$ manifold, as a function on R^3 . Given such topological data, one must find a classical solution that minimizes the energy of the chiral Lagrangian. An order of magnitude estimate for the size $L_{Skymion}$ of a Skymion gives

$$L_{Skymion} \sim \frac{1}{f_\pi e_{Skyme}} , \quad (5)$$

which is independent of large N_c .

However, let us pose and consider whether this construction really makes sense. This solitonic picture says that baryons can be regarded as coherent states of Goldstone bosons of QCD. Although the latter are special due to the simple and universal origin and also due to the light mass, they are one of many varieties of bi-quark mesons. In particular, there are known and experimentally measured cubic couplings between pions and heavier spin one mesons, such ρ mesons. A condensate of pions, as in a Skymion, would show up as a source term for a ρ meson equation of motion and ρ must also have its own coherent state excited. In turn, this will disturb the conventional Skymion picture and modify it quantitatively. This is a clear signal that the usual Skymion picture of the baryon has to be modified significantly in the context of full QCD.

Perhaps because of this, and perhaps for other reasons, the picture of baryon as Skymion have produced mixed results when compared to experimental data. In this note, we will explore how this problem is partially cured, in a natural and simple manner without new unknown parameter, and how the resulting baryons look qualitatively and quantitatively different from that of Skymion. As we will see, the holographic picture naturally brings a gauge-principle in the bulk description of the flavor dynamics in such a way that all spin one mesons as well as pions would enter the construction of baryons on the equal footing. The basic concept of baryon as coherent states of mesons would remain unchanged, however. It is the purpose of this note to outline this new approach to baryons and to explore the consequences.

2 A Holographic QCD

A holographic QCD is similar to the chiral perturbation theory in the sense that we deal with exclusively gauge-invariant operators of the theory. The huge difference is, however, that this new approach tends to treat all gauge-invariant objects together. Not only the light meson fields like pions but also heavy vector mesons and baryons appear together, at least in principle. In other words, a holographic QCD deals with all color-singlets simultaneously, giving us a lot more predictive power. Later we will see examples of this more explicitly.

This new approach is motivated by the large N_c limit of gauge theories [2] and in particular by the AdS/CFT correspondence [5]. One of the more interesting notion that emerged in this regard over the last three decades is the concept of *the master field*. The idea is that in the large N limits of matrix theories with a gauge symmetry, the gauge-singlet observables behaves semiclassically in the large N limit [6]. Probably the most astounding twist is the emergence of a new spatial direction in such a picture. As we learned from AdS/CFT, *the master fields* have to be thought of not as four-dimensional fields but at least five-dimensional, with the additional direction being labeled by energy scale. We refer to this new direction as the holographic direction.

The standard AdS/CFT duality gives us a precise equivalence between the large N_c maximally supersymmetric Yang-Mills theories and the type IIB string theory or IIB supergravity in $\text{AdS}_5 \times S^5$. Here, *the master fields* are nothing but closed string fields such as the gravity multiplet and excited closed string fields. It is also believed that such a duality extends to other large N field theories such as ordinary QCD which is neither supersymmetric nor conformal. The question is then how to find the right dual theory of the large N_c QCD.

One set of ideas for this, dubbed bottom-up [7], is similar in spirit to the chiral perturbation theory. One assumes that an approximate conformal symmetry exists for a wide range of energy scales and build up a bulk gravity theory coupled to more bulk fields, as would be dictated by the AdS/CFT rules if QCD were conformal. The conformal symmetry is subsequently broken by cutting off the geometry at both the infrared and the ultraviolet and by introducing boundary conditions. Necessary degrees of freedoms, namely the master fields, are introduced as needed by construction, rather than derived, and in this sense the approach is similar to the conventional chiral perturbation theory.

The other approach is referred to as top-down, and here one tries to realized the QCD as a low energy limit of some open string theory on D-branes, from which a holographic model follows as the closed string theory dual. Arguably, the best model of this kind we know of is the D4-D8 system, where $U(N_c)$ D4 gauge theory compactified on a thermal circle provides large N_c Yang-Mills sector. The $U(N_F)$ gauge theory on D8 brane, on the other hand, can be thought of bi-quark meson

sector in the adjoint representation of the $U(N_F)$ flavor symmetry. A crucial aspect of this model, although to be expected from general AdS/CFT principles, is that the vector-like flavor symmetry is promoted to a gauge theory in the bulk. This D4-D8 model was slowly developed over the years, starting with Witten’s initial identification of the dual geometry for D4 branes wrapped on a thermal circle [8], study of glueball mass spectra of pure QCD without matter [9, 10], the introduction of mesons via D8 branes [11], and very recent study of baryons as solitonic objects [12, 13, 14] on D8 branes. In this section, we will review glueballs and mesons in this D4-D8 model.

2.1 Holographic Pure QCD from D4

The story starts with a stack of D4 branes which is compactified on a circle. The circle here is sometimes called “thermal” in that one requires anti-periodic boundary condition on all fermions, just as one would for the Euclidean time circle when studying finite temperature field theory. The purpose of having a spatial “thermal” circle is to give mass to the fermionic superpartners and thus break supersymmetry. As is well known, the low energy theory on N D p branes is a maximally supersymmetric $U(N)$ Yang-Mills theory in $p + 1$ dimensions, so putting N_c D4 branes on a thermal circle, we obtain pure $U(N_c)$ Yang-Mills theory in the noncompact $3 + 1$ dimensions. We are interested in large N_c limit, so the $U(1)$ part can be safely ignored, and we may pretend that we are studying $SU(N_c)$ theory instead. While the anti-periodic boundary condition generates massgap only to fermionic sector at tree level, scalar partners also become massive since there is no symmetry to prohibit their mass any more. Only the gauge multiplet is protected.

We then extrapolate the general idea of AdS/CFT to this non-conformal case, which states that, instead of studying strongly coupled large N_c Yang-Mills theory, one may look at its dual closed string theory. The correct closed string background to use is nothing but the string background generated by the D4 branes in question. This geometry was first written down by Gibbons and Maeda [15] in the 1980’s, and later reinterpreted by Witten in 1998 as the dual geometry for D4 branes on a thermal circle [8]. The metric is most conveniently written as

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu + f(U) d\tau^2) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \quad (6)$$

with $R^3 = \pi g_s N_c l_s^3$ and $f(U) = 1 - U_{KK}^3/U^3$. The topology of the spacetime is $R^{3+1} \times D \times S^4$, with the coordinate τ labeling the azimuthal angle of the disk D , with $\tau = \tau + \delta\tau$ and $\delta\tau = 4\pi R^{3/2}/(3U_{KK}^{1/2})$. The circle parameterized by τ is the thermal circle. The dilaton is

$$e^{-\Phi} = \frac{1}{g_s} \left(\frac{R}{U}\right)^{3/4}, \quad (7)$$

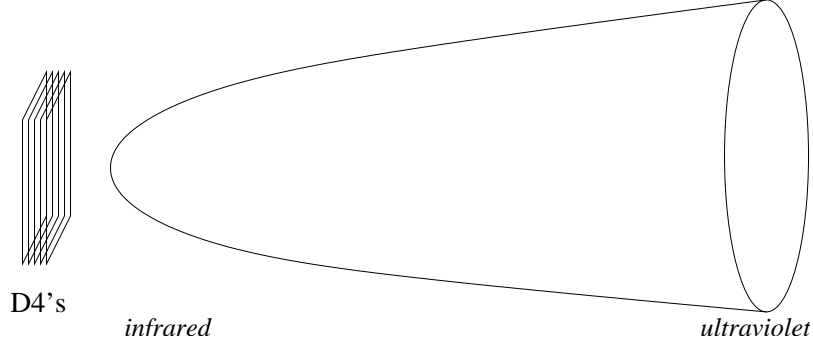


Figure 1: A schematic diagram showing the dual geometry. A stack of D4's responsible for the dual geometry are shown for an illustrative purpose, although the actual spacetime does not include them. The manifold shown explicitly is spanned by the angle τ and the radial coordinate U . The thermal circle spanned by τ closes itself in the infrared end due to the strong interaction of QCD. Small excitations of metric (and its multiplet) at the infrared end correspond to glueballs.

while the antisymmetric Ramond-Ramond background field C_3 is such that dC_3 carries N_c unit of flux along S^4 .

In the limit of large curvature radius, thus large N_c , and in the limit of large 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c$, the duality collapse to a relationship between the theory of D4 branes to type IIA supergravity defined in this background. Given the lack of useful method of string theory quantization in curved background, this is the best we can do at the moment. Therefore, all computations in any of holographic QCD must assume such a limit and extrapolate to realistic regime at the end of the computation. This is also the route that we follow in this note.

Among remarkable works in early days of AdS/CFT is the study of glueball spectra in this background [9, 10]. They considered small fluctuations of IIA gravity multiplet in the above background, with the plane-wave like behavior along x^μ and L^2 normalizability along the remaining six directions. They identified each of such modes as glueballs up to spin 2, and computed their mass² eigenvalues as dictated by the linearized gravitational equation of motion.

This illustrates what is going on here. We can think of the duality here as a simple statement that the open string side and the closed string side is one and the same theory. The reason we have apparently more complicated description on the open string side is because there we started with a misleading and redundant set of elementary fields, namely the gauge field whose number scales as N_c^2 , only to be off-set by the gauge symmetry. The closed string side, or its gravity limit, happens to be more smart about what are the right low energy degrees of freedom and encodes only gauge-invariant ones. For pure Yang-Mills theory like this, the only

gauge-invariant objects are glueballs, so the dual gravitational side should compute the glueball physics.

The expectation that there exists a more intelligent theory consisting only of gauge-invariant objects in the large N_c limit is thus realized via string theory in a somewhat surprising manner that *the master fields*, those truly physical degrees of freedom, actually live not in four dimensional Minkowskian world but in five or higher dimensional curved geometry. This is not however completely unanticipated, and was heralded in the celebrated work by Eguchi and Kawai in early 1980's [16] which is all the more remarkable in retrospect. For the rest of this note, we will continue this path and try to incorporate massless quarks to the story.

2.2 Adding Mesons via D4-D8 Complex

To add mesons, Sakai and Sugimoto introduced the N_F D8 branes, which share the coordinates x^μ with the above D4 branes [11] and are transverse to the thermal circle τ . Before we trade off the N_c D4 branes in favor of the dual gravity theory, this would have allowed massless quark as open strings ending on both the D4 and the D8 branes. As the D4's are replaced by the dual geometry, however, the 4-8 open strings have to be paired up into 8-8 open strings, which are naturally identified as bi-quark mesons. From the viewpoint of D8 branes, the lightest of such mesons belong to a $U(N_F)$ gauge field.

The $U(N_F)$ gauge theory on D8 branes has the action

$$-\frac{4\pi^2 l_s^4 \mu_8}{8} \int \sqrt{-h_{8+1}} e^{-\Phi} \text{tr} \mathcal{F}^2 + \mu_8 \int C_3 \wedge \text{tr} e^{2\pi\alpha' \mathcal{F}}, \quad (8)$$

where the contraction is via the induced metric of D8 and $\mu_p = 2\pi/(2\pi l_s)^{p+1}$ with $l_s^2 = \alpha'$. The induced metric on the D8 brane is

$$h_{8+1} = \frac{U^{3/2}(w)}{R^{3/2}} (dw^2 + \eta_{\mu\nu} dx^\mu dx^\nu) + \frac{R^{3/2}}{U^{1/2}(w)} d\Omega_4^2, \quad (9)$$

after we trade off the holographic (or radial) coordinate U in favor of a conformal one w as

$$w = \int_{U_{KK}}^U R^{3/2} dU' / \sqrt{U'^3 - U_{KK}^3}, \quad (10)$$

which resides in a finite interval of length $\sim O(1/M_{KK})$ where $M_{KK} \equiv 3U_{KK}^{1/2}/2R^{3/2}$. Thus, the topology of the D8 worldvolume is $R^{3+1} \times I \times S^4$. The nominal Yang-Mills coupling g_{YM}^2 is related to the other parameters as

$$g_{YM}^2 = 2\pi g_s M_{KK} l_s, \quad (11)$$

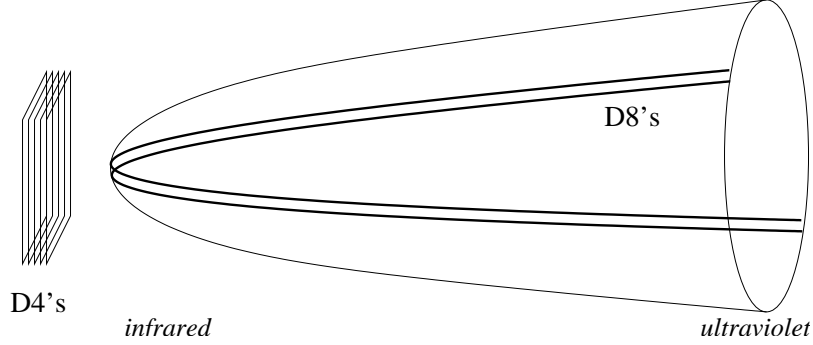


Figure 2: The figure shows how D8's are added to the system. Low energy excitations (also located at the infrared end) of D8-D8 open strings are bi-quark mesons.

which is not, however, a physical parameter on its own. The low energy parameters of this holographic theory are M_{KK} and λ , which together with N_c sets all the physical scales such as the QCD scale and the pion decay constant.

In the low energy limit, we ignore S^4 direction on which D8's are completely wrapped, and find a five-dimensional Yang-Mills theory with a Chern-Simons term

$$-\frac{1}{4} \int_{4+1} \frac{1}{e(w)^2} \sqrt{-h_{4+1}} \text{tr} \mathcal{F}^2 + \frac{N_c}{24\pi^2} \int_{4+1} \omega_5(\mathcal{A}), \quad (12)$$

where the position-dependent Yang-Mills coupling of this flavor gauge theory is

$$\frac{1}{e(w)^2} = \frac{e^{-\Phi} V_{S^4}}{2\pi(2\pi l_s)^5} = \frac{\lambda N_c}{108\pi^3} M_{KK} \frac{U(w)}{U_{KK}} \quad (13)$$

with V_{S^4} the position-dependent volume of S^4 . The Chern-Simons coupling with $d\omega_5(\mathcal{A}) = \text{tr} \mathcal{F}^3$ arises because $\int_{S^4} dC_3 \sim N_c$.

As advertised, this by itself generates many of bi-quark mesons of QCD. More specifically, all of vector and axial-vector mesons and the pion multiplet are encoded in this five-dimensional $U(N_F)$ gauge field. The vector mesons and the axial vector mesons are more straightforward conceptually, since any “compactification” of five-dimensional Yang-Mills theory would lead to an infinite tower of four-dimensional massive vector fields. Although the radial direction w (or U) is infinite in terms of proper length, equation of motion is such that normalizable fields are strongly pushed away from the boundary, making it effectively a compact direction. The usual Kaluza-Klein reduction (in the somewhat illegal but convenient axial gauge $\mathcal{A}_w = 0$),

$$\mathcal{A}_\mu(x; w) = i\alpha_\mu(x)\psi_0(w) + i\beta_\mu(x) + \sum_n a_\mu^{(n)}(x)\psi_{(n)}(w) \quad (14)$$

contains an infinite number of vector fields, whose action can be derived explicitly as,

$$\int dx^4 \mathcal{L} = \int dx^4 \sum_n \text{tr} \left\{ \frac{1}{2} \mathcal{F}_{\mu\nu}^{(n)} \mathcal{F}^{\mu\nu(n)} + m_{(n)}^2 a_\mu^{(n)} a^{\mu(n)} \right\} + \dots, \quad (15)$$

with $\mathcal{F}_{\mu\nu}^{(n)} = \partial_\mu a_\nu^{(n)} - \partial_\nu a_\mu^{(n)}$. The ellipsis denotes zero mode part, to be discussed shortly, as well as infinite number of couplings among these infinite varieties of mesons, all of which come from the unique structure of the five-dimensional $U(N_F)$ Yang-Mills Lagrangian in (12). Because \mathcal{A} has a specific parity, the parity of a_n 's are determined by the parity of the eigenfunctions $\psi_{(n)}(w)$ along the fifth direction. Since the parity of any one-dimensional eigenvalue system alternates, an alternating tower of vector and axial-vector fields emerge as the masses $m_{(n)}$ of the KK modes increase.

For each such eigenmode, five-dimensional massless vector field has three degrees of freedom, so is natural for massive four-dimensional vector fields to appear. An exception to this naive counting, which is specific to the gauge theory, is the zero mode sector. In Eq. (14), we separated it out from the rest as $\alpha(x)$ and $\beta(x)$ terms. To understand this part, it is better to give up the axial gauge and consider the Wilson line,

$$U(x) = e^{i \int_w \mathcal{A}(x,w)}, \quad (16)$$

which, as the notation suggests, one identifies with the pion field $U(x) = e^{2i\pi(x)/f_\pi}$. Upon taking a singular gauge transformation back to $\mathcal{A}_w = 0$, one finds that it is related to α and β as

$$\alpha_\mu(x) \equiv \{U^{-1/2}, \partial_\mu U^{1/2}\}, \quad 2\beta_\mu(x) \equiv [U^{-1/2}, \partial_\mu U^{1/2}]. \quad (17)$$

Truncating to this zero mode sector reproduces a Skyrme Lagrangian of pions [4] as a dimensional reduction of the five-dimensional Yang-Mills action,

$$\int dx^4 \left(\frac{f_\pi^2}{4} \text{tr} (U^{-1} \partial_\mu U)^2 + \frac{1}{32e_{Skyrme}^2} \text{tr} [U^{-1} \partial_\mu U, U^{-1} \partial_\nu U]^2 \right), \quad (18)$$

with $f_\pi^2 = (g_{YM}^2 N_c) N_c M_{KK}^2 / 54\pi^4$ and $1/e_{Skyrme}^2 \simeq 61(g_{YM}^2 N_c) N_c / 54\pi^7$. No other quartic term arise, nor do we find higher order terms in derivative, although we do recover the Wess-Zumino-Witten term from the Chern-Simons term [11]. To compare against actual QCD, we must fix $\lambda = g_{YM}^2 N_c \simeq 17$ and $M_{KK} \simeq 0.94 \text{ GeV}$ to fit both the pion decay constant f_π and the mass of the first vector meson. After this fitting, all other infinite number of masses and coupling constants are fixed. This version of the holographic QCD is extremely predictive.

Let us emphasize that the meson system here comes with a qualification. Note that we treated D8 branes differently than D4 branes. The latter are replaced by the dual geometry while the former are kept as branes. This has to be because we are

interested in objects charged under $U(N_F)$, whereas we are only interested in singlets under $U(N_c)$. However, we not only treated D8 as branes but also as probe branes, meaning that the backreaction of D8 to the dual geometry of D4's is ignored. In terms of field theory language, we effectively ignored Feynman diagrams involving quarks in the internal lines, resulting in the quenched approximation.

3 Holographic Baryons

The baryon can be naturally regarded as a coherent state of mesons in the large N_c . In the conventional chiral Lagrangian approach, is the Skyrmon made from pions, which we argued cannot be the full picture. In D4-D8 model of holographic QCD above, especially, pions are only zero mode part of a holographic flavor theory, and infinite towers of vector and axial-vector mesons are packaged together with pions into a single five-dimensional $U(N_F)$ gauge field. This suggests that the picture of baryon as a soliton must be lifted to a five-dimensional soliton of this $U(N_F)$ gauge theory in the bulk, in such a manner that spin one mesons contribute to construction of baryons as well. In this section, we explore classical and quantum properties of this holographic and new version of Skyrmon.

3.1 The Instanton Soliton

The five-dimensional effective action for the $U(N_F)$ gauge field in Eq. (12) admits solitons which carry a Pontryagin number

$$\frac{1}{8\pi^2} \int_{R^3 \times I} \text{tr} F \wedge F = k, \quad (19)$$

with integral k . We denoted by F the non-Abelian part of \mathcal{F} (and similarly later, \mathcal{A} for non-Abelian part of \mathcal{A}). The smallest unit with $k = 1$ turns out to carry quantum numbers of the baryon. The easiest way to see this identification is to relate it to the Skyrmon [4] of the chiral perturbation theory.

Recall that both instantons and Skyrmons are labeled by the third homotopy group π_3 of a group manifold, which is the integer for any semi-simple Lie group manifold G . For the Skyrmon, the winding number show up in the classification of maps

$$U(x) : R^3 \rightarrow SU(N_F). \quad (20)$$

For the instanton whose asymptotic form is required to be pure gauge,

$$A(x, w \rightarrow \pm\infty) = ig_{\pm}(x)^{\dagger} dg_{\pm}(x), \quad (21)$$

the winding number is in the classification of the map

$$g_{-}(x)^{\dagger} g_{+}(x) : R^3 \rightarrow SU(N_F). \quad (22)$$

The relationship between the two types of the soliton is immediate [18]. Recall that the U field of chiral perturbation theory is obtained in our holographic picture as the open and infinite Wilson line along w direction. On the other hand, the Wilson line computes nothing but $g_-(x)^\dagger g_+(x)$, so we find that

$$U_k(x) = e^{i \int_w A^{(k)}(x,w)} \quad (23)$$

carries k Skyrmon number exactly when $A^{(k)}$ carries k Pontryagin number. Therefore, the instanton soliton in five dimensions is the holographic image of the Skyrmons in four dimensions. We will call it the instanton soliton.

Normal instantons on a conformally flat four-manifold are well studied, and the counting of zero modes says that for k instanton in $U(N_F)$ theory, there are $4kN_F$ collective coordinates. For the minimal case with $k = 1$ and $N_F = 2$, giving us eight collective coordinates. They are four translations, one overall size, and three gauge rotations. For our instanton solitons, this counting does not hold any more.

Unlike the usual Yang-Mills theory in trivial R^4 background, the effective action has a position-dependent inverse Yang-Mills coupling $1/e(w)^2$ which is a monotonically increasing function of $|w|$. Since the Pontryagin density contributes to action as multiplied by $1/e(w)^2$, this tends to position the soliton near $w = 0$ and also shrink it for the same reason. The F^2 energy of a trial configuration with size ρ can be estimated easily in the small ρ limit,^{#2}

$$E_{\text{Pontryagin}} = \frac{\lambda N_c}{27\pi} M_{KK} \times \left(1 + \frac{1}{6} M_{KK}^2 \rho^2 + \cdots \right), \quad (24)$$

which clearly shows that the energy from the kinetic term increases with ρ . This by itself would collapse the soliton to a point-like one, making further analysis impossible.

A second difference comes from the presence of the additional Chern-Simons term $\sim \text{tr} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F}$, whereby the Pontryagin density $F \wedge F$ sources some of the gauge field \mathcal{A} minimally. This electric charge density costs the Coulombic energy

$$E_{\text{Coulomb}} \simeq \frac{1}{2} \times \frac{e(0)^2 N_c^2}{10\pi^2 \rho^2} + \cdots, \quad (25)$$

again in the limit of $\rho M_{KK} \ll 1$. This Coulombic energy tends to favor larger soliton size, which competes against the shrinking force due to $E_{\text{Pontryagin}}$.

The combined energy is minimized at [12, 13, 14]

$$\rho_{\text{baryon}} \simeq \frac{(2 \cdot 3^7 \cdot \pi^2 / 5)^{1/4}}{M_{KK} \sqrt{\lambda}}, \quad (26)$$

^{#2} The estimate of energy here takes into account the spread of the instanton density $D(x^i, w) \sim \rho^4 / (r^2 + w^2 + \rho^2)^4$, but ignores the deviation from the flat geometry along the four spatial directions.

and the classical mass of the stabilized soliton is

$$\begin{aligned}
m_B^{classical} &= (E_{\text{Pontryagin}} + E_{\text{Coulomb}}) \Big|_{\text{minimum}} \\
&= \frac{\lambda N_c}{27\pi} M_{KK} \times \left(1 + \frac{\sqrt{2 \cdot 3^5 \cdot \pi^2 / 5}}{\lambda} + \dots \right). \quad (27)
\end{aligned}$$

As was mentioned above, the size ρ_{baryon} is significantly smaller than $\sim 1/M_{KK}$. We have a classical soliton whose size is a lot smaller than the fundamental scale of the effective theory. On the other hand, this small soliton size is still much larger than its own Compton size $1/m_B^{classical} \simeq 27\pi/(M_{KK}\lambda N_c)$, justifying our assertion that this is indeed a soliton.

Note that the instanton soliton size is much smaller than the Skyrmon size when the 't Hooft coupling is large.^{#3} We already saw that the Skyrmon size is determined by the ratio of the two dimensionful couplings in the chiral Lagrangian. Using the values of these coupling derived from our D4-D8 model, the would-be Skyrmon size is

$$L_{\text{Skyrmion}} \sim \frac{1}{f_\pi e_{\text{Skyrme}}} \sim \frac{1}{M_{KK}}. \quad (28)$$

On the other hand, the size of the holographic baryon is

$$\rho_{baryon} \sim \frac{1}{M_{KK}\sqrt{\lambda}}. \quad (29)$$

The difference is substantial in the large 't Hooft coupling limit where this holographic QCD makes sense. Why is this?

Simply put, the Skyrmon solution of size $\sim 1/M_{KK}$ is a bad approximation, because it solves the chiral Lagrangian which neglects all other spin one mesons. This truncation can be justified for processes involving low energy pions. The baryon is, however, a heavy object and contains highly excited modes of pions, and will excite relatively light vector mesons as well since U is coupled to vector and axial-vector mesons nontrivially at cubic level. Therefore, the truncation to the pion sector is not a good approximation as far as solitonic baryons are concerned, especially for large 't Hooft coupling constant.^{#4} We emphasize this difference because many of existing computation of the baryon physics based on the Skyrmon picture must be thus

^{#3}One must not confuse these solitonic sizes with the electromagnetic size of baryons. The latter is dictated by how photons interact with the baryon, and in the holographic QCD with $\lambda \gg 1$ is determined at ρ meson scale and independent of λ , due to the vector dominance. One may think of these solitonic sizes as being hadronic.

^{#4}There were previous studies that incorporated the effect of coupling a single vector meson, namely the lightest ρ meson, on the Skyrmon which showed a slight shrinkage of the soliton [19] as we would have expected in retrospect.

rethought in terms of the new instanton soliton picture. We will consider implication of this new picture of the baryon in next sections.

Our solitonic picture of the baryon has a close tie to the usual AdS/CFT picture of baryons as wrapped D-branes. A D4 brane wrapped along the compact S^4 corresponds to a baryon vertex on the five-dimensional spacetime [11], as follows from an argument originally due to Witten [20]. To distinguish them from the D4 branes supporting QCD, let us call them D4'. On the D4' worldvolume we have again a Chern-Simons coupling of the form,

$$\mu_4 \int C_3 \wedge 2\pi\alpha' d\mathcal{A}' \quad (30)$$

with D4' gauge field \mathcal{A}' , which can be evaluated over S^4 as

$$2\pi\alpha'\mu_4 \int_{S^4 \times R} dC_3 \wedge \mathcal{A}' = N_c \int_R \mathcal{A}' , \quad (31)$$

where R denotes the worldline in the noncompact part of the spacetime. This shows that the background dC_3 flux over S^4 induces N_c unit of the electric charge. On the other hand, the Gauss constraint for \mathcal{A}' demands that the net charge should be zero, so the wrapped D4' can exist only if N_c end points of fundamental strings are attached to D4' to cancel this charge. In turn, the other ends of the fundamental strings must go somewhere, and the only place it can go is D8 branes. One can think of these strings as individual quarks that constitute the baryon. Also, because of these fundamental strings, the wrapped D4' cannot be separated from D8's without a lot of energy cost. The lowest energy state would be one where D4' is on top of D8's, which then would smear out as an instanton. The latter is exactly the instanton soliton of ours.

3.2 Quantum Numbers

For the sake of simplicity, and also because the quarks in this model have no bare mass, we will take $N_F = 2$ for the rest of the note. A unit instanton soliton in question comes with six collective coordinates. Three correspond to the position in R^3 , and three correspond to the gauge angles in $SU(N_F = 2)$. If the soliton is small enough ($\rho M_{KK} \ll 1$), there exists approximate symmetries $SO(4) = SU(2)_+ \times SU(2)_-$ at $w = 0$, so the total rotational symmetry of a small solution at origin is $SU(N_F = 2) \times SU(2)_+ \times SU(2)_-$. Let us first see how the quantized instanton soliton fit into representations of this approximate symmetry group.

The instanton can be rotated by an conjugate $SU(2)$ action as,

$$F \rightarrow S^\dagger F S , \quad (32)$$

with any 2×2 special unitary matrices S which span \mathbf{S}^3 .^{#5} Then, the quantization of the soliton is a matter of finding eigenstates of free and nonrelativistic nonlinear sigma-model onto \mathbf{S}^3 [21]. S itself admits an $SO(4)$ symmetry of its own,

$$S \rightarrow USV^\dagger. \quad (33)$$

Because of the way the spatial indices are locked with the gauge indices, these two rotations are each identified as the gauge rotation, $SU(N_F = 2)$, and half of the spatial rotations, say, $SU(2)_+$. Eigenstates on \mathbf{S}^3 are then nothing but the familiar angular momentum eigenfunctions of three Euler angles, conventionally denoted as

$$|s : p, q\rangle. \quad (34)$$

Recall that the quadratic Casimirs of the two $SU(2)$'s (associated with U and V rotations) always coincide to be $s(s+1)$. One can proceed exactly in the same manner for anti-instantons, where $SU(2)_+$ is replaced by $SU(2)_-$.

Therefore, under $SU(N_F = 2) \times SU(2)_+ \times SU(2)_-$, the quantized instantons are in [23]

$$(2s+1; 2s+1; 1), \quad (35)$$

while the quantized anti-instantons are in

$$(2s+1; 1; 2s+1). \quad (36)$$

Possible values for s are integers and half-integers. However, we are eventually interested in $N_c = 3$, in which case spins and isospins are naturally half-integral. Thus we will subsequently consider the case of fermionic states only. Exciting these isospin come at energy cost. See Hata et.al. [13] for mass spectra of some excited instanton solitons.

4 Holographic Dynamics

The solitonic baryon is a coherent object which is made up of pions as well as of vector and axial-vector mesons. This implies that the structure of the soliton itself contains all the information on how the baryon interacts with these infinite tower of mesons. This sort of approach has been also used [17] in the Skyrme picture of old days, where, for instance, the leading axial coupling for a nucleon emitting a soft pion was computed following such thoughts. The difference here is that, instead of just pions, all spin one mesons enter this holographic construction of the baryon, and this enables us to compute all low energy meson-hadron vertices simultaneously.

^{#5}Since S and $-S$ rotates the solution the same way the moduli space is naively \mathbf{S}^3/Z_2 . However at quantum level, we must consider states odd under this Z_2 as well, so the moduli space is \mathbf{S}^3 .

4.1 Dynamics of Hairy Solitons: Generalities

First, we would like to illustrate the point by considering another kind of solitons. The magnetic monopoles [22] appear as solitons in non-Abelian Yang-Mills theories spontaneously broken to a subgroup containing a $U(1)$ factor, such as in $SU(2) \rightarrow U(1)$, and carries a magnetic charge. Usually it is a big and fluffy object and must be treated as a classical object. However, if we push the electric Yang-Mills coupling to be large enough, so that the magnetic monopole size become comparable or even smaller than the symmetry breaking scale, we have no choice but to treat it as a point-like object. The effective action for this monopole field \mathcal{M} (spinless for example) should contain at least,

$$\left| \left(\partial_\mu + i \frac{4\pi}{e} \tilde{A}_\mu \right) \mathcal{M} \right|^2, \quad (37)$$

where \tilde{A} is the dual photon of the unbroken $U(1)$ gauge field. We know this coupling exists simply because the monopole has the magnetic charge $4\pi/e$. But how do we know the latter fact? Because the soliton solution itself exhibits a long range magnetic Coulomb tail of the form

$$F^{monopole} \sim \frac{4\pi}{e} \frac{1}{r^2}. \quad (38)$$

If we replace the solitonic monopole by the quanta of the field \mathcal{M} but do not couple to the dual photon field as above, we would end up with a local excitation. However, a magnetic monopole (or an electrically charge particle) is not really a local object. Creating one always induces the corresponding long range magnetic (electric) Coulomb field. To ensure that the effective field theory represent the magnetic monopole accurately, we must make sure that creating a quanta of \mathcal{M} is always followed by creation of the necessary magnetic Coulomb field. This is achieved by coupling the local field \mathcal{M} to the gauge field \tilde{A} at an appropriate strength. This is a somewhat unconventional way to understand the origin of the minimal coupling of the monopole to the dual gauge field \tilde{A} .

4.2 The Small Size Matters

Before going further, let us briefly pose and ask about the validity of such an approach for our solitonic baryon. The key to this is a set of inequalities among three natural scales that enter the baryon physics, which are

$$\frac{1}{M_{KK}} \gg \frac{1}{M_{KK}\sqrt{\lambda}} \gg \frac{1}{M_{KK}N_c\lambda}. \quad (39)$$

They hold in the large N_c and large λ limit. The first is the length scale of mesons, the second is the classical size of the solitonic baryon, and the third is the Compton wavelength of the baryon since its mass is $\sim M_{KK}N_c\lambda$.

The first inequality tells us that the baryon tends to be much smaller than mesons and thus can be regarded almost pointlike when interacting with mesons. This justifies the effective field theory approach where we think of each baryon as small excitation of a field. One does this precisely when the object in question can be treated as if it has no internal structure other than quantum numbers like spins.

The second inequality tells us that the quantum uncertainty associated with the baryon is far smaller than the classical core size of the soliton. This is important because, otherwise, one may not be able to trust anything about the classical features of the soliton at quantum level. When the second inequality holds, it enables us to make use of the classical shape of the soliton and to extract information about how meson interact with the baryon. The fact we have a small soliton size and an even smaller Compton size of that soliton is very fortunate.

4.3 Holographic Dynamics of Baryons

As with the small magnetic monopole case, we wish to trade off the (quantized) instanton soliton in favor of local baryon field(s) and make sure to encode the long-range tails of the soliton in how the baryon field(s) interacts with the low energy gauge fields. Our instanton soliton has two types of distinct but related long-range field. The first is due to the Pontryagin density and goes like

$$F_{mn} \sim \frac{\rho_{baryon}^2}{(r^2 + w^2)^2} , \quad (40)$$

while the second is the Coulomb field due to the Chern-Simons coupling between \mathcal{A} and $F \wedge F$,

$$\mathcal{F}_{0n} \sim \frac{e(w)^2 N_c}{(r^2 + w^2)^{3/2}} . \quad (41)$$

The latter is the five-dimensional analog of the electric Coulomb tail.

Apart from the fact that we have two kinds of long-range fields, there is another important difference from the monopole case. As we saw in section 3.2, the solitonic baryon has \mathbf{S}^3 worth of internal moduli, quantization of which gave us the various spin/isospin baryons. Since the gauge direction of the magnetic long range field is determined by coordinate on \mathbf{S}^3 , the field strengths associated with the Pontryagin density should be smeared out by quantum fluctuation along the moduli space. It is crucial for our purpose that what we mean by long-range fields of the instanton soliton are actually these quantum counterpart, not the naive classical one. Basic features of the smearing out effect and relevant identities can be found in next subsection.

The electric Coulomb tail should be encoded in a minimal coupling to the Abelian part of \mathcal{A} . For a spin/isospin half Baryon, \mathcal{B} , we anticipate a minimal term of the form

$$\bar{\mathcal{B}}(N_c \mathcal{A}_m^{U(1)} + A_\mu) \gamma^m \mathcal{B} . \quad (42)$$

This is uniquely fixed by the Coulomb charge N_c and the $SU(N_F = 2)$ representation of the quantized instanton. The purely magnetic tail of the soliton is more subtle to deal with. From the simple power counting, it is obvious that the coupling responsible for such a tail must have one higher dimension than the minimal coupling, hinting at the field strength F of the $SU(N_F = 2)$ part coupling directly to a baryon bilinear, such as

$$\bar{\mathcal{B}} F_{mn} \gamma^{mn} \mathcal{B} . \quad (43)$$

It turns out that this is precisely the right structure to mimic the long-range magnetic fields of quantized instantons and anti-instantons.^{#6}

To show that the latter vertex is indeed the precisely right one, one must consider the following points. (1) Is this the unique term that can reproduce the correct quantum-smeared long-range instanton and anti-instanton tail? (2) If so, how do we fix the coefficient, taking into account the quantum effects. (3) And is the estimate reliable? The answers are long and technical. We refer the readers to literatures [12, 14, 23] for precise answers to these questions, but here state that the answers are all affirmative and that the effective action of mesons and baryons is uniquely determined by this simple consideration. This is true at least in the large N_c and the large λ limit.

This leads to the following five-dimensional effective action,

$$\begin{aligned} & \int d^4x dw \left[-i\bar{\mathcal{B}} \gamma^m D_m \mathcal{B} - im_{\mathcal{B}}(w) \bar{\mathcal{B}} \mathcal{B} + \frac{2\pi^2 \rho_{baryon}^2}{3e^2(w)} \bar{\mathcal{B}} \gamma^{mn} F_{mn} \mathcal{B} \right] \\ & - \int d^4x dw \frac{1}{4e^2(w)} \text{tr} \mathcal{F}_{mn} \mathcal{F}^{mn} , \end{aligned} \quad (44)$$

with the covariant derivative given as $D_m = \partial_m - i(N_c \mathcal{A}_m^{U(1)} + A_m)$ with A_m in the fundamental representation of $SU(N_F = 2)$.

The position-dependent mass $m_{\mathcal{B}}(w) \sim 1/e(w)^2$ is a very sharp increasing function of $|w|$, such that in the large N_c and large λ limit, the baryons wavefunction is effectively localized at $w = 0$. This is the limit where the above effective action is trustworthy. We find

$$\frac{2\pi^2 \rho_{baryon}^2}{3e^2(0)} = \frac{N_c}{\sqrt{30}} \cdot \frac{1}{M_{KK}} , \quad (45)$$

so the last term involving baryons can be actually dominant over the minimal coupling, despite that it looks subleading in the derivative expansion. As it turns out, this term is dominant for cubic vertex processes involving pions or axial vector mesons, whereas the minimal coupling dominates for those involving vector mesons [14].

^{#6} In fact, a prototype of this simple method makes a brief appearance in the landmark work on Skyrmion by Adkins, Nappi, and Witten [17]. In their case, however, this gives only the pion-baryon interactions, forcing them to a related but somewhat different formulation. In our case, this method generates all meson-baryon interactions, however.

4.4 Basic Identities and Isospin-Dependence

We have discussed general ideas behind the effective action approach and given the explicit results for isospin 1/2 case. The only term that is not obvious is the coupling between baryons and the field strength F , with the coefficient $2\pi^2\rho_{baryon}^2/3e^2(0)$, and we would like to spend a little more time on its origin. Apart from convincing readers that the derivation of the effective action is actually rigorous, this would also allow us to outline how the result generalizes for higher isospin baryons, such Δ particles, as well.

Each and every quantum of the baryon field \mathcal{B} is supposed to represent a quantized (anti-)instanton soliton. Let us recall that the quantization of the soliton involves finding wavefunctions on the moduli space of the soliton, which is S^3 . Since the moduli encode the gauge direction of the instanton soliton, the classical gauge field is quantum mechanically smeared and should be replaced by its expectation values as

$$F \rightarrow \langle\langle S^\dagger F S \rangle\rangle = \langle\langle \Sigma_{ab} \rangle\rangle F^b, \quad (46)$$

with $2\Sigma_{ab} \equiv \text{tr} [\tau_a S^\dagger \tau_b S]$. $\langle\langle \cdots \rangle\rangle$ means taking expectation value on wavefunctions on the moduli space of the soliton, and computes the quantum smearing effect.

The effective action (44) would make sense if and only if each quanta of the baryon field \mathcal{B} is equipped with precisely the right smeared-out gauge field of this type. How is this possible? For the simplest case of isospin 1/2, the relevant identity that shows this reads^{#7}

$$\langle\langle 1/2 : p', q' | \Sigma_{ab} | 1/2 : p, q \rangle\rangle = -\frac{1}{3} (\mathcal{U}(1/2 : p', q')^{\epsilon'}_{\beta'})^* \sigma_a^{\beta'\beta} \tau_b^{\epsilon'\epsilon} \mathcal{U}(1/2 : p, q)_\beta^\epsilon \quad (47)$$

where $\mathcal{U}(1/2 : p, q)$ is the two-component spinor/isospinor of $J_3 = p$, $I_3 = q$, and $J^2 = I^2 = 3/4$. Identifying the two-component spinor \mathcal{U} as the upper half of the four-component spinor \mathcal{B} representing positive energy states, one can show that the equation of motion for the gauge field coupled to \mathcal{B} is

$$(\nabla \cdot F)_m^a \sim \nabla_n (\bar{\eta}_{nm}^b \mathcal{U}^\dagger (\sigma_b \tau^a) \mathcal{U}) + \cdots, \quad (48)$$

which shows, via (47), that the quanta \mathcal{U} of \mathcal{B} would be accompanied by the correctly smeared long range tail of gauge field of type (46). The right hand side comes from the coupling of type

$$\bar{\mathcal{B}} F \mathcal{B} \quad (49)$$

in (44). A similar match can be shown for negative energy states, where the 't Hooft symbol $\bar{\eta}$ is replaced by η and \mathcal{U} by its anti-particle counterpart \mathcal{V} . A careful check

^{#7} This identity for $s = 1/2$ is originally due to Adkins, Nappi, and Witten, who obtained it in the context of the Skyrmion. The moduli space of a Skyrmion and that of our instanton soliton coincides, so the same identity holds.

of the normalization leads us to the coefficient $2\pi^2\rho_{baryon}^2/3e^2(0)$, where the number 3 in the denominator came from the factor $1/3$ in Eq. (47).

It turns out that this goes beyond $s = 1/2$. The identity (47) is generalized to for arbitrary half-integral s as [23]

$$\langle\langle s : p', q' | \Sigma_{ab} | s : p, q \rangle\rangle = -\frac{s}{s+1} \cdot (\mathcal{U}(s : p', q')^{\epsilon'_1 \epsilon'_2 \dots \epsilon'_{2s}}_{\beta' \alpha_2 \dots \alpha_{2s}})^* \sigma_a^{\beta' \beta} \tau_b^{\epsilon' \epsilon} \mathcal{U}(s : p, q)^{\epsilon \epsilon_2 \dots \epsilon_{2s}}_{\beta \alpha_2 \dots \alpha_{2s}}, \quad (50)$$

where the left-hand-side is again evaluated as wavefunction-overlap integral on the moduli space S^3 of the instanton soliton. \mathcal{U} is now that of higher spin/isospin field with symmetrized multi-spinor/multi-isospinor indices. As with $\mathcal{U}(1/2)$, $\mathcal{U}(s)$'s are positive energy spinors with each index taking values 1 and 2. This implies a cubic interaction term of type

$$\bar{\mathcal{B}}_s F \mathcal{B}_s \quad (51)$$

where \mathcal{B}_s denotes a local baryon field of isospin s and $SO(4) = SU(2)_+ \times SU(2)_-$ angular momentum $[s]_+ \otimes [0]_- \oplus [0]_+ \otimes [s]_-$. Relative to the isospin $1/2$ case, the coefficient is increased from $1/3$ to $s/(s+1)$, which reflects the obvious fact that higher angular momentum states would be less and less smeared.

Finally, with $s > 1/2$ baryons included, there are one more type of processes allowed where a baryon changes its own isospin by emitting isospin 1 mesons. The relevant identities for these processes are

$$\langle\langle s : p', q' | \Sigma_{ab} | s+1 : p, q \rangle\rangle = -\frac{1}{2} \sqrt{\frac{2s+1}{2s+3}} \cdot [\mathcal{U}(s : p', q')^\dagger \mathcal{U}(s+1 : p, q)_{ab}] , \quad (52)$$

where 3×3 spin/isospin s wavefunctions $\mathcal{U}(s+1 : p, q)_{ab}$ are

$$(\mathcal{U}(s+1 : p, q)_{ab})^{\epsilon_1 \dots \epsilon_{2s}}_{\alpha_1 \dots \alpha_{2s}} \equiv (\sigma_2 \sigma_a)^{\beta \beta'} (\tau_2 \tau_b)_{\epsilon \epsilon'} \mathcal{U}(s+1 : p, q)^{\epsilon \epsilon' \epsilon_1 \dots \epsilon_{2s}}_{\beta \beta' \alpha_1 \dots \alpha_{2s}} . \quad (53)$$

This shows up in the effective action of baryon as a coupling of type

$$\bar{\mathcal{B}}_{s+1} F \mathcal{B}_s . \quad (54)$$

The complete effective action of baryons with such arbitrary half-integer isospins was given in Ref. [23]. For the rest of the note, we will confine ourselves to isospin $1/2$ case.

5 Nucleons

Nucleons are the lowest lying baryons with isospin and spin $1/2$. As such, they arise from the isospin $1/2$ holographic baryon field \mathcal{B} whose effective action is given explicitly above. This effective action contains interaction terms between currents of

\mathcal{B} with the $U(N_F)$ gauge field of five dimensions, and thus contain an infinite number of interaction terms between nucleons and mesons, specifically all cubic couplings involving nucleons emitting pion, vector mesons, or axial-vector mesons. Extracting four-dimensional amplitudes of interests is a simple matter of dimensional reduction from $R^{3+1} \times I$ to R^{3+1} . In this section, we show this procedure, showcase some of the simplest examples such comparisons, and comment on how the results should be taken in view of various approximation schemes we relied on.

5.1 Nucleon-Meson Effective Actions

The effective action for the four-dimensional nucleons is derived from this, by identifying the lowest eigenmode of \mathcal{B} upon the KK reduction along w direction as the proton and the neutron. Higher KK modes would be also isospin half baryons, but the gap between the ground state and excited state is very large in the holographic limit, so we consider only the ground state. We mode expand $\mathcal{B}_\pm(x^\mu, w) = \mathcal{N}_\pm(x^\mu) f_\pm(w)$, where \pm refers to the chirality along w direction, and reconstitute a four-dimensional spinor \mathcal{N} with $\gamma^5 \mathcal{N}_\pm = \pm \mathcal{N}_\pm$ as its chiral and anti-chiral components. The lowest KK eigenmodes $f_\pm(w)$ solve

$$\left[-\partial_w^2 \mp \partial_w m_{\mathcal{B}}(w) + (m_{\mathcal{B}}(w))^2 \right] f_\pm(w) = m_{\mathcal{N}}^2 f_\pm(w), \quad (55)$$

with some minimum eigenvalue $m_{\mathcal{N}} > m_{\mathcal{B}}(0) = m_{\mathcal{B}}^{classical}$. This nucleon mass $m_{\mathcal{N}}$ will generally differ from the five-dimensional soliton mass $m_{\mathcal{B}}^{classical}$, due to quantization of light modes such as spread of the wavefunction $f_{L,R}$ along the fifth direction.

Inserting this into the action (44), we find the following structure of the four-dimensional nucleon action

$$\int dx^4 \mathcal{L}_4 = \int dx^4 \left(-i \bar{\mathcal{N}} \gamma^\mu \partial_\mu \mathcal{N} - i m_{\mathcal{N}} \bar{\mathcal{N}} \mathcal{N} + \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{axial}} \right), \quad (56)$$

where we have, schematically, the vector-like couplings

$$\mathcal{L}_{\text{vector}} = -i \bar{\mathcal{N}} \gamma^\mu \beta_\mu \mathcal{N} - \sum_{k \geq 0} g_V^{(k)} \bar{\mathcal{N}} \gamma^\mu a_\mu^{(2k+1)} \mathcal{N}, \quad (57)$$

and the axial couplings to axial mesons,

$$\mathcal{L}_{\text{axial}} = -\frac{i g_A}{2} \bar{\mathcal{N}} \gamma^\mu \gamma^5 \alpha_\mu \mathcal{N} - \sum_{k \geq 1} g_A^{(k)} \bar{\mathcal{N}} \gamma^\mu \gamma^5 a_\mu^{(2k)} \mathcal{N}. \quad (58)$$

All the couplings constants $g_{V,A}^{(k)}$ and g_A are calculated by suitable wave-function overlap integrals involving f_\pm and $\psi_{(n)}$'s.

Although we did not write so explicitly, isospin triplet mesons and singlet mesons have different coupling strength to the nucleons, so there are actually two sets of couplings $(g_A, g_A^{(k)}, g_V^{(k)})$, one for isosinglet mesons, such as ω and η' , and the other for isotriplet mesons, such as ρ and π . The leading contribution to axial couplings in the isospin triplet channel arise from the direct coupling to F_{mn} , and are all proportional to ρ_{baryon}^2 . All the rest are dominated by terms from the five-dimensional minimal coupling to \mathcal{A}_m . We refer interested readers to Ref. [12, 14] for explicit form of these coupling constants.

5.2 Numbers and Comments

To showcase typical predictions from the above setup, let us quote two notable examples for the nucleons [14]. The first is the cubic coupling of the lightest vector mesons to the nucleon, to be denoted as $g_{\rho NN}$ for the isotriplet meson ρ and $g_{\omega NN}$ for the iso-singlet meson ω . In the above effective action, these two are denoted collectively as $g_V^{(0)}$. An interesting prediction of this holographic effective action of nucleons is that

$$\frac{g_{\omega NN}}{g_{\rho NN}} = N_c + \delta , \quad (59)$$

where the leading N_c is a consequence from the five-dimensional minimal coupling to \mathcal{A} while the subleading correction δ arises from the direct coupling to the field strength F . With $N_c = 3$ and $\lambda \simeq 17$ (the latter is required by fitting f_π and M_{KK} to the pion decay constant and the vector meson masses to actual QCD), we find

$$\frac{g_{\omega NN}}{g_{\rho NN}} \simeq 3 + 0.6 = 3.6 . \quad (60)$$

Extracting ratios like this from experimental data is somewhat model-dependent, with no obvious consensus, but the ratio is believed to be larger than 3 and numbers around 4-5 are typically found. Given the crude nature of our approximation and that there is no tunable parameter other than the QCD scale and f_π , the agreement is uncanny. A more complete list of various cubic couplings between spin one mesons and nucleons has been worked out in Ref. [14] and further elaborated recently in Ref. [24].

The leading axial coupling to pions, g_A , is somewhat better measured at $\simeq 1.26$. Our prediction is [12]

$$g_A = \frac{2\lambda N_c (\rho_{baryon} M_{KK})^2}{81\pi^2} + \dots = \left(\frac{24}{5\pi^2} \right)^{1/2} \times \frac{N_c}{3} + \dots , \quad (61)$$

where the leading term arise from the direct coupling to the field strength F and the ellipsis denotes the subleading and higher correction. While this does not look too

good, we must remember that this holographic model is effectively a quenched QCD, missing out on possible $O(1)$ corrections. From old studies of large N_c constituent models, a group theoretical $O(1)$ correction has been proposed for this type of operators, which states that the next leading correction would amounts to $N_c \rightarrow N_c + 2$ ^{#8}. So, in a more realistic version where we take into account of the backreaction of D8 branes on the dual geometry, we may anticipate for $N_c = 3$

$$g_A \simeq \left(\frac{24}{5\pi^2} \right)^{1/2} \times \frac{N_c + 2}{3} + O(1/N_c) \simeq 1.16 + O(1/N_c) . \quad (62)$$

Finally, $O(1/N_c)$ is partly captured by the minimal coupling term our quenched model, which turns out to give roughly a 10% positive correction, making the total very close to the measured quantity 1.26.

These two illustrate nicely what kind of predictions can be made and how accurate their predictions can be when compared to experimental data. Much more rich array of predictions exist, such as other cubic couplings between mesons and baryons, anomalous magnetic moment [12], complete vector dominance of electromagnetic form factors [14], and detailed prediction on momentum dependence of such form factors [25, 26].

However, one should be a bit more cautious. The model, as an approximation to real QCD, has many potential defects. The main problem is that all of this is in the context of large N_c and that any prediction, such as above two, has to involve an extensive extrapolation procedure. Many ambiguities can be found in such a procedure, and we chose a particular strategy of computing all quantities and analytically continuing the final expressions for the amplitudes to realistic QCD regime. The fact it works remarkably well does not really support its validity in any rigorous sense. Also the D4-D8 model we employed include many massive fields which are not part of ordinary four-dimensional QCD, and one should be cautious in using the holographic QCD for physics other than simple low energy processes.

Despite such worries, the D4-D8 holographic QCD turned out to be far better than one may have anticipated. We have shown how it accommodates not only the (vector) meson sector but the baryon sector very competently.^{#9} Whether or not the holographic QCD can be elevated to a controlled and justifiable approximation to real QCD remains to be seen, depending crucially on having a better understanding of the string theory in the curved spacetime. Nevertheless, it is fair to say that we finally have a rough grasp of the physics that controls *the master fields*, and perhaps

^{#8}See Ref. [14] for more explanations and references

^{#9} One of the acutely missing story is how the spinless mesons (except Goldstone bosons) would fit in the story. Initial investigation of this gave a possibly disappointing result, although it may have more to do with how the lightest scalar mesons are rather complicated objects and may not be a bi-quark meson of conventional kind [27].

this insight by itself will lead to a better and more practical formulation of the QCD in the future.

6 Electromagnetic Properties

Holographic baryons and their effective action in the bulk also encodes how baryons, and in particular, nucleons would interact with electromagnetism. For this, one follows the usual procedure of AdS/CFT where operators in the field theory are matched up with non-normalizable modes of bulk fields. Operationally, one simply introduces the boundary photon field \mathcal{V} as a nonnormalizable mode, which adds to β -term in the expansion of \mathcal{A} ,

$$A_\mu(x; w) = i\alpha_\mu(x)\psi_0(w) + \mathcal{V}_\mu(x) + i\beta_\mu(x) + \sum_n a_\mu^{(n)}(x)\psi_{(n)}(w) , \quad (63)$$

and repeat the dimensional reduction to the four dimensions. For instance, computation of anomalous magnetic moments of proton and neutron can be done with relative ease, and gives remarkably good agreement with measured values [14].

For more detailed accounts of electromagnetic properties of baryon, we refer the readers to Refs. [14, 25, 26]. Here we will only consider the most notable feature of the electromagnetic properties, namely the complete vector dominance, whereby all electromagnetic interactions are entirely mediated by the infinite tower of vector mesons. This also illustrate well how the holographic QCD can give a sweeping and qualitative prediction and also where it could fail.

The vector dominance means that there is no point-like charge, which, in view of the minimal coupling between \mathcal{A} and \mathcal{B} in (44), sounds pretty odd. To understand what's going on, one must consider quadratic structures in the vector meson sector. Defining

$$\zeta_k = \int dw \frac{1}{2e(w)^2} \psi_{(2k+1)}(w) , \quad (64)$$

for parity even eigenfunctions $\psi_{(2k+1)}$'s, the quadratic part of the vector meson is [11]

$$\sum_k \text{tr} \left[-\frac{1}{2} |dv^{(k)}|^2 - m_{(2k+1)}^2 |v^{(k)} - \zeta_k(\mathcal{V} + i\beta)|^2 \right] , \quad (65)$$

where we introduced the shifted vector fields

$$v^{(k)} = a^{(2k+1)} + \zeta_k(\mathcal{V} + i\beta) . \quad (66)$$

This mixing of vector mesons and photon is at the heart of the vector dominance. (The axial-vector mesons, $a^{(2k)}$'s, do not mix with photon because of the parity.)

Now let us see how this mixing of vector fields enters the coupling of baryons with electromagnetic vector field \mathcal{V} . Taking the minimal coupling, we find

$$\int dw \bar{\mathcal{B}} \gamma^m A_m \mathcal{B} = \bar{B} \gamma^\mu \mathcal{V}_\mu B + \sum_k g_{V,min}^{(k)} \bar{B} \gamma^\mu a_\mu^{(2k+1)} B + \dots, \quad (67)$$

where the ellipsis denotes axial couplings to axial vectors as well as coupling to pions via α_μ and β_μ . $g_{V,min}^{(k)}$ is the cubic coupling between k -th vector meson and the baryon, or more precisely its leading contribution coming from the minimal coupling to \mathcal{A} . Again, the presence of the direct minimal coupling to the photon \mathcal{V} seems to contradict the notion of vector dominance. However, it is advantageous to employ the canonically normalized vector fields $v^{(k)}$ in place of $a^{(k)}$, upon which this becomes

$$\bar{B} \gamma^\mu \mathcal{V}_\mu B + \sum_k g_{V,min}^{(k)} \bar{B} \gamma^\mu (v_\mu^{(k)} - \zeta_k \mathcal{V}_\mu) B + \dots. \quad (68)$$

On the other hand,

$$\begin{aligned} \sum_k g_{V,min}^{(k)} \zeta_k &= \sum_k \int dw' |f_+(w')|^2 \psi_{(2k+1)}(w') \times \int dw \frac{1}{2e(w)^2} \psi_{(2k+1)}(w) \\ &= \int dw' |f_+(w')|^2 \times \int dw \delta(w - w') = 1, \end{aligned} \quad (69)$$

where we made use of the definite parities of $1/e(w)^2$ and $\psi_{(n)}$'s and also of the completeness of $\psi_{(n)}$'s. This sum rule $\sum_k g_{V,min}^{(k)} \zeta_k = 1$ forces

$$\bar{B} \gamma^\mu \mathcal{V}_\mu B + \sum_k g_{V,min}^{(k)} \bar{B} \gamma^\mu (v_\mu^{(k)} - \zeta_k \mathcal{V}_\mu) B + \dots = \sum_k g_{V,min}^{(k)} \bar{B} \gamma^\mu v_\mu^{(k)} B + \dots \quad (70)$$

and the baryon couples to the photon field \mathcal{V} only via $v^{(k)}$'s which mixes with \mathcal{V} in their mass terms.

This choice of basis is only for the sake of clarity. Regardless of the basis, the above shows that no coupling between \mathcal{V} and \mathcal{B} can occur in the infinite momentum limit. This statement is clear in the $\{\mathcal{V}; v^{(k)}\}$ basis which is diagonal if the mass term is negligible. Alternatively, we can ask for the invariant amplitude of the charge form factor, to which the minimal coupling contributes [14]

$$F_{1,min}(q^2) = 1 - \sum_k \frac{g_{V,min}^{(k)} \zeta_k q^2}{q^2 + m_{(2k+1)}^2} = \sum_k \frac{g_{V,min}^{(k)} \zeta_k m_{(2k+1)}^2}{q^2 + m_{(2k+1)}^2} \quad (71)$$

with the momentum transfer q . For small momentum transfer, the first few light vector mesons dominate the form factors by mediating between the baryon and the

photon. This end fit with experimental data pretty well. Similar computation can be done for the magnetic form factor, from which one also finds the (anomalous) magnetic moment that fits the data pretty well [14, 12].

However, for large momentum transfer, the form factor decays as $1/q^2$ which is actually too slow for real QCD baryons. Estimates based on the parton picture say that the decay should be $\sim 1/q^{2(N_c-1)}$. This dramatic failure of the form factor for large momentum regime should not be a big surprise. The theory we started with is a low energy limit of D4-D8 complex compactified (with warp factors) on $S^1 \times S^4$. As such, one has to truncate infinite number of massive modes in order to reach a QCD-like theory in the boundary and must stay away from that cut-off scale to be safe from this procedure. For large momentum transfers, say larger than M_{KK} , the computation we relied on has no real rationale. This should caution readers that the holographic QCD, at least in the limited forms that are available now, is not a fix for everything. One really must view it as a vastly improved version of the chiral Lagrangian approach, with many hidden symmetries now manifest, but still suitable only for low energy physics.

7 More Comments

D4-D8 holographic model of QCD is the most successful model of its kind known. It reproduces in particular detailed particle physics of mesons and baryons. One reason for its success can be found in the fact that it builds on the the meson sector, the lightest of which is lighter than the natural cut-off scale M_{KK} . Apart from $1/N_c$ and $1/\lambda$ expansions imposed by general AdS/CFT ideas, one also must be careful with low energy expansion as well, because, as we stated before, the model includes many more massive Kaluza-Klein modes and even string modes that are not part of ordinary QCD. For low energy processes, nevertheless, one would hope that these extra massive states (above M_{KK}) do not contribute too much, which seems to be the case for low lying meson sector [11].

Our solitonic and holographic model of baryons elevates the classic Skyrme picture based on pions to a unified model involving all spin one mesons in addition to pions. This is why the picture is extremely predictive. As we saw in this note, for low momentum processes, such as soft pion processes, soft rho meson exchanges, and soft elastic scattering of photons, the model's predictions compare extremely well with experimental data. It is somewhat mysterious that the baryon sector works out almost as well as the meson sector, since baryons are much heavier than M_{KK} in the large N_c and the large λ limit.

Note that the soliton underlying the baryon is nearly self-dual in the large λ limit. For instance, Eq. (27) shows that the leading, would-be BPS, mass is dominant over the rest by a factor of λ . There must be a sense in which the soliton is approximately

supersymmetric with respect to the underlying IIA string theory, even though the background itself breaks all supersymmetry at scale M_{KK} . One may argue that even though there are many KK modes and even stringy modes lying between the naive cut-off scale M_{KK} and the baryon mass scale $M_{KK}N_c\lambda$, these non-QCD degrees of freedom would be paired into approximate supermultiplets, reducing their potentially destructive effect, especially because the baryon itself is roughly BPS. Whether or not one can actually quantify such an idea for the model we have is unclear, but if possible it would be an important step toward rigorously validating holographic approaches to baryons in this D4-D8 set-up.

There are more work to be done. One important direction is to perform more refined comparisons against experiments. In particular, extracting coupling constants from raw data seems quite dependent on theoretical models, and it is important to compute directly measurable amplitudes starting from the effective action of ours. Nucleon-nucleon scattering amplitudes or more importantly the nucleon-nucleon potential would be a good place to start [28, 29, 24]. Another profitable path would be to consider dense system such as neutron stars as well as physics of light nuclei, where our model with far less tunable parameters would give unambiguous predictions. This will in turn further test the model as well.

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